

# Next-to-leading order corrections to capacity for nondispersive nonlinear optical fiber channel in intermediate power region

A. A. Panarin,<sup>1,\*</sup> A. V. Reznichenko,<sup>2,1,†</sup> and I. S. Terekhov<sup>2,1,‡</sup>

<sup>1</sup>*Novosibirsk State University, Novosibirsk, 630090 Russia*

<sup>2</sup>*Budker Institute of Nuclear Physics of Siberian Branch Russian Academy of Sciences, Novosibirsk, 630090 Russia*

We consider the optical fiber channel modelled by the nonlinear Shrödinger equation with zero dispersion and additive Gaussian noise. Using Feynman path-integral approach for the model we find corrections to conditional probability density function, output signal distribution, conditional and output signal entropies, and the channel capacity at large signal-to-noise ratio. We demonstrate that the correction to the channel capacity is positive for large signal power. Therefore, this correction increases the earlier calculated capacity for a nondispersive nonlinear optical fiber channel in the intermediate power region.

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## I. INTRODUCTION.

The problem of information transmission through a noisy communication channel is considered more than 60 years. First results of the solution of the problem were obtained by Shannon Ref. [1]. In Ref. [1] Shannon introduced the channel capacity  $C$ , which gives the maximum amount of information that can be reliably transmitted over a noisy communication channel. For the first time he obtained the logarithmic dependence of the capacity on signal power for a linear communication channel with additive Gaussian noise:

$$C \propto \log(1 + \text{SNR}), \quad (1)$$

where  $\text{SNR} = P/N$  is the signal-to-noise power ratio,  $P$  is the signal power, and  $N$  is the noise power. It means that in order to increase the capacity one has to increase the signal power  $P$  for the fixed noise power  $N$ . There is a question: how the nonlinearity in a communication channel affects the result (1). The interest to the nonlinear channels started to increase when the fiber optics communication system began intensively developing. It is connected with the Kerr nonlinearity in optical fibers. The influence of nonlinearity on capacity is investigated both for dispersive and nondispersive optical channels. The channels with dispersion were studied in numerous papers, see, e.g., [2–12] and references therein. Despite the fact that the capacity for the nonlinear channel with dispersion was considered in many papers the exact in nonlinearity result is still not found due to difficulty of the problem. Therefore as the first step in understanding of the effects of nonlinearity impact in the channel one can consider the nonlinear channel with zero average dispersion. The nonlinear nondispersive optical fiber channels are also considered in numerous papers, see, e.g.,

[13–18]. Of course, the problem of capacity calculation for these channels is simpler than the problem with dispersion. However it is still quite a challenging problem especially at large parameter SNR, and new techniques and methods are highly desirable to advance these studies [3, 16, 17, 19–21].

The channel capacity  $C$  can be determined as the maximum of the mutual information  $I_{P_X[X]}$  with respect to the probability density function (PDF)  $P_X[X]$  of an input signal  $X$ :

$$C = \max_{P_X[X]} I_{P_X[X]}. \quad (2)$$

The maximum value of the mutual information  $I_{P_X[X]}$  in Eq. (2) should be found at the given average signal power:

$$P = \int \mathcal{D}X P_X[X] |X|^2. \quad (3)$$

The PDF  $P_X[X]$  also obeys the normalization condition:

$$\int \mathcal{D}X P_X[X] = 1, \quad (4)$$

that fixes the integration measure  $\mathcal{D}X = d\text{Re}X d\text{Im}X$ . The mutual information is defined as the difference of output signal entropy  $H[Y]$  and conditional entropy  $H[Y|X]$ :

$$I_{P_X[X]} = H[Y] - H[Y|X], \quad (5)$$

where the entropies are defined as

$$H[Y|X] = - \int \mathcal{D}X \mathcal{D}Y P_X[X] P[Y|X] \log P[Y|X], \quad (6)$$

$$H[Y] = - \int \mathcal{D}Y P_{\text{out}}[Y] \log P_{\text{out}}[Y], \quad (7)$$

here  $P_{\text{out}}[Y]$  is output signal PDF:

$$P_{\text{out}}[Y] = \int \mathcal{D}X P_X[X] P[Y|X], \quad (8)$$

\*Electronic address: panarin.a.a@mail.ru

†Electronic address: a.v.reznichenko@inp.nsk.su

‡Electronic address: i.s.terekhov@gmail.com

and  $P[Y|X]$  is conditional probability density function, i.e., the PDF to have the output signal  $Y$  when the input signal is  $X$ . The measure  $\mathcal{D}Y$  is defined as  $\int \mathcal{D}Y P[Y|X] = 1$ . Our definitions (5)–(7) imply that we measure the capacity in units  $(\log 2)^{-1}$  bit per symbol (also known as nat per symbol). Usually the input and output signals are the functions of time which have certain bandwidth. Therefore the sampling of the temporal signal should be introduced to define discrete-time memoryless channel. In this case the capacity should be proportional to bandwidth. But we discuss nondispersive channels. It means that we can consider the functions  $X(t)$  and  $Y(t)$  at the same time moment and calculate only per-sample (i.e., for one time elementary channel) quantities.

To calculate the mutual information we should know the conditional probability density function  $P[Y|X]$  for the channel. This quantity depends on the channel model. As was mentioned above for the nondispersive channel the temporal signal waveform changes during propagation independently for every time moment. Therefore, instead of consideration of the evolution of  $\psi(z, t)$  we can consider a set of parallel independent scalar channels [13, 17], the so-called per-sample channels. We choose the signal propagation model described by the following equation, see [17]:

$$\partial_z \psi(z) - i\gamma |\psi(z)|^2 \psi(z) = \eta(z), \quad (9)$$

i.e., the nonlinear Shrödinger equation with zero dispersion and with additive noise  $\eta(z)$ . In Eq. (9)  $\psi(z)$  is the complex function which describes the signal propagation in the channel,  $\gamma$  is Kerr nonlinearity parameter, the function  $\eta(z)$  describes the additive noise in the channel. The noise has the zero mean  $\langle \eta(z) \rangle_\eta = 0$  and the correlation function  $\langle \eta(z) \bar{\eta}(z') \rangle_\eta = Q \delta(z - z')$ , where  $Q$  is the noise power per unit length. The function  $\psi(z)$  obeys the boundary condition  $\psi(0) = X$ . In our notations the per-sample signal power and noise power are  $P$  and  $N = QL$ , respectively, where  $L$  is the signal propagation length. Here the signal power  $P$  is defined in Eq. (3). For the channel (9) the conditional PDF  $P[Y|X]$ , i.e., the probability density to receive the signal  $\psi(L) = Y$  when  $\psi(0) = X$ , was found in the form of infinite series [13, 16] within Martin-Siggia-Rose formalism based on the quantum field theory methods [22, 23]. Using the obtained probability  $P[Y|X]$  the lower bound for the channel capacity at large  $\text{SNR} = P/(QL)$  was found:

$$C \geq \frac{\log(\text{SNR})}{2} + \frac{1 + \gamma_E - \log(4\pi)}{2} + \mathcal{O}\left(\frac{\log(\text{SNR})}{\text{SNR}}\right), \quad (10)$$

where  $\gamma_E \approx 0.5772$  is the Euler constant. The first term in the right-hand side of the inequality (10) was obtained in Ref. [16], whereas the second term was obtained in Ref. [18]. One can see that the lower bound (10) of the capacity grows as  $(1/2) \log(\text{SNR})$  instead of  $\log \text{SNR}$ . The factor  $1/2$  appears due to the loss of information about the phase of the signal, see Ref. [17]. In Ref. [18]

the new method of calculation of the conditional PDF  $P[Y|X]$  was developed. This method allowed us to sum the infinite series for  $P[Y|X]$  obtained in Refs. [13, 16] at large SNR, and to obtain the simple form of the conditional PDF  $P[Y|X]$  in the leading order in  $1/\text{SNR}$ , see Ref. [18]. In Ref. [18] using this form of  $P[Y|X]$  we calculated the capacity of the nonlinear nondispersive optical fiber channel in the intermediate power region

$$QL \ll P \ll (Q\gamma^2 L^3)^{-1} \quad (11)$$

with the accuracy  $\mathcal{O}(QL/P) + \mathcal{O}(\gamma^2 QPL^3)$ . Moreover, it was shown that at sufficiently large power  $P$  in the region

$$(\gamma L)^{-1} \ll P \ll (Q\gamma^2 L^3)^{-1} \quad (12)$$

the found capacity is greater than the bound (10), but in the region the capacity grows only as  $\log \log P$  with increasing of signal power  $P$  instead  $(1/2) \log \text{SNR}$ , see Eq. (54) in Ref. [18]. However at  $P \gg (Q\gamma^2 L^3)^{-1}$  the capacity should be of the order of  $(1/2) \log(P/QL)$ . It means that we have to understand how one asymptotical regime for the capacity transforms to another one. To this end we should calculate the first nonzero corrections in parameter  $QL$ . Moreover, to clarify the accuracy of the results obtained in Ref. [18] we also should find the first nonzero correction to the channel capacity which is proportional to the noise power  $QL$ . To calculate the correction to the channel capacity  $C$  we should know the corrections of this order to the conditional PDF  $P[Y|X]$ , entropies (6)–(7), and the optimal input signal distribution  $P_{\text{opt}}[X]$ .

The paper is organized in the following way. In Sec. II we present the results of calculations of the next-to-leading order correction to the conditional PDF  $P[Y|X]$ . In this Section we briefly remind the method of  $P[Y|X]$  calculation developed in details in Ref. [18]. The result of the calculation of the output signal distribution  $P_{\text{out}}[Y]$  in the next-to-leading order concludes Section II. Sec. III is devoted to the calculation of the conditional entropy  $H[Y|X]$  and the output signal entropy  $H[Y]$  in the next-to-leading order in  $1/\text{SNR}$ . In Sec. IV we present the calculation of the optimal input signal distribution  $P_{\text{opt}}[X]$ , and in Sec. V using the obtained expression for  $P_{\text{opt}}[X]$  we find the correction to the capacity (2). We discuss our results in Sec. VI.

## II. CALCULATION OF THE CONDITIONAL PDF $P[Y|X]$ AND OUTPUT SIGNAL PDF $P_{\text{out}}[Y]$ AT LARGE SNR

### A. Method for the conditional PDF $P[Y|X]$ calculation

This section is based on the method described in details in Ref. [18], therefore here we just schematically describe the calculation. We start our consideration from

the expression for the conditional PDF  $P[Y|X]$  in the path-integral form [16, 23, 24] in retarded discretization scheme, see, e.g., Supplemental Materials of Ref. [21] or Ref. [18]:

$$P[Y|X] = \int_{\psi(0)=X}^{\psi(L)=Y} \mathcal{D}\psi \exp \left\{ -\frac{S[\psi]}{Q} \right\}, \quad (13)$$

where the effective action  $S[\psi]$  reads

$$S[\psi] = \int_0^L dz \left| \partial_z \psi - i\gamma |\psi|^2 \psi \right|^2.$$

In the case when the parameter  $\text{SNR} \gg 1$  it is convenient to rewrite the form (13) in the following way, see Ref. [21]:

$$P[Y|X] = \Lambda e^{-\frac{S[\Psi_{cl}(z)]}{Q}}, \quad (14)$$

where the normalization factor is

$$\Lambda = \int_{\tilde{\psi}(0)=0}^{\tilde{\psi}(L)=0} \mathcal{D}\tilde{\psi} e^{-\frac{S[\Psi_{cl}(z) + \tilde{\psi}(z)] - S[\Psi_{cl}(z)]}{Q}}, \quad (15)$$

and the function  $\Psi_{cl}(z)$  is the “classical” solution of the equation  $\delta S[\Psi_{cl}] = 0$ , where  $\delta S$  is the variation of the action  $S[\psi]$ . The equation for the function  $\Psi_{cl}$  can be written in the form:

$$\frac{d^2 \Psi_{cl}}{dz^2} - 4i\gamma |\Psi_{cl}|^2 \frac{d\Psi_{cl}}{dz} - 3\gamma^2 |\Psi_{cl}|^4 \Psi_{cl} = 0, \quad (16)$$

with the boundary conditions  $\Psi_{cl}(0) = X$ ,  $\Psi_{cl}(L) = Y$ . To calculate the conditional probability we should calculate the exponent contribution and the path-integral in Eq. (14).

We start our calculation from exponent  $e^{-\frac{S[\Psi_{cl}(z)]}{Q}}$ . Since we calculate the function  $P[Y|X]$  with the accuracy  $1/\text{SNR}$  we should find the solution of Eq. (16) with this accuracy. Following Ref. [18] we find such solution linearizing Eq. (16) in the vicinity of the solution  $\Psi_0(z)$  of the channel equation (9) with zero noise. The function  $\Psi_0(z)$  reads

$$\Psi_0(z) = \rho \exp \left\{ i\mu \frac{z}{L} + i\phi^{(X)} \right\}, \quad (17)$$

where  $\mu = \gamma L |X|^2$ . Note that the solution (17) is also solution of Eq. (16) but it satisfies only the input boundary condition  $\Psi_0(0) = X = \rho e^{i\phi^{(X)}}$ , where  $\rho = |X|$ . Therefore, to fulfill the output boundary condition  $\Psi_{cl}(L) = Y$  we look for the solution of Eq. (16) in the form

$$\Psi_{cl}(z) = (\rho + \varkappa(z)) \exp \left\{ i\mu \frac{z}{L} + i\phi^{(X)} \right\}, \quad (18)$$

where the function  $\varkappa(z)$  is assumed to be small:  $|\varkappa(z)| \ll \rho$ . In Ref. [18] we argued that statistically significant for  $P[Y|X]$  functions  $\varkappa(z)$  are at least of the order of  $\sqrt{Q}$ . The equation for the function  $\varkappa$  has the form, see Eq. (79) in Ref. [18]:

$$\begin{aligned} \frac{d^2 \varkappa}{dz^2} - 2i\frac{\mu}{L} \frac{d\varkappa}{dz} - 4\frac{\mu^2}{L^2} \text{Re}[\varkappa] &= 4i\frac{\mu}{L\rho} (\varkappa + \bar{\varkappa}) \frac{d\varkappa}{dz} + \frac{\mu^2}{L^2\rho} [5\varkappa^2 + 10|\varkappa|^2 + 3\bar{\varkappa}^2] + \\ \frac{|\varkappa|^2\mu}{L^2\rho^2} \left[ 4iL \frac{d\varkappa}{dz} + 9\mu\bar{\varkappa} + 14\mu\varkappa \right] &+ \frac{3\mu^2}{L^2\rho^2} \varkappa^3 + \frac{3\mu^2}{L^2\rho^3} |\varkappa|^2 [3|\varkappa|^2 + 2\varkappa^2] + \frac{3\mu^2}{L^2\rho^4} |\varkappa|^4 \varkappa. \end{aligned} \quad (19)$$

The boundary conditions for  $\varkappa$  are as follows:

$$\varkappa(0) = 0, \quad \varkappa(L) = Y e^{-i\phi^{(X)} - i\mu} - \rho \equiv x_0 + iy_0, \quad (20)$$

Since the  $|\varkappa| \ll \rho$  we can solve Eq. (19) using perturbation theory in the parameter  $\varkappa/\rho$  and present the solution  $\varkappa$  in the form

$$\varkappa(z) = \varkappa_1(z) + \varkappa_2(z) + \varkappa_3(z) + \dots \quad (21)$$

The functions  $\varkappa_1(z) \propto \sqrt{Q}$  and  $\varkappa_2(z) \propto Q$  were found in Ref. [18]: see Eqs. (82), (86), and (87) therein. The equation for the function  $\varkappa_3(z)$  can be easily obtained from Eq. (19). The equation for the function  $\varkappa_3(z)$  and

the solution of this equation are cumbersome, therefore, we do not present them here. But we present the final result  $S[\Psi_{cl}]$  in the leading  $S_1$ , next-to-leading  $S_2$ , and next-to-next-to-leading order  $S_3$  in parameter  $1/\sqrt{\text{SNR}}$ :

$$S[\Psi_{cl}] = S_1 + S_2 + S_3 + \mathcal{O}(\text{SNR}^{-5/2}), \quad (22)$$

where

$$S_1 = \frac{(1 + 4\mu^2/3)x_0^2 - 2\mu x_0 y_0 + y_0^2}{L(1 + \mu^2/3)}, \quad (23)$$

$$S_2 = \frac{\mu/\rho}{135L(1+\mu^2/3)^3} \left\{ \mu(4\mu^4 + 15\mu^2 + 225)x_0^3 + (23\mu^4 + 255\mu^2 - 90)x_0^2y_0 + \mu(20\mu^4 + 117\mu^2 - 45)x_0y_0^2 - 3(5\mu^4 + 33\mu^2 + 30)y_0^3 \right\}, \quad (24)$$

$$S_3 = \frac{\mu^2}{2100L(\mu^2+3)^5\rho^2} \left[ x_0^4(148\mu^8 - 12345\mu^6 - 24570\mu^4 - 806085\mu^2 + 396900) - 12\mu x_0^3y_0(901\mu^6 + 9990\mu^4 + 84105\mu^2 - 139860) + 36\mu x_0y_0^3(385\mu^6 + 6198\mu^4 + 30165\mu^2 + 8820) - 6x_0^2y_0^2(980\mu^8 + 11857\mu^6 + 24210\mu^4 - 350595\mu^2 - 49140) + 3y_0^4(700\mu^8 + 8365\mu^6 + 23826\mu^4 - 32535\mu^2 - 34020) \right]. \quad (25)$$

Since  $x_0$  and  $y_0$  are of the order of  $\sqrt{Q}$  (see the text after Eq. (17) in Ref. [18]) one can see that  $S_1/Q$ ,  $S_2/Q$ , and  $S_3/Q$  are of the order of  $(\text{SNR})^0$ ,  $\text{SNR}^{-1/2}$ , and  $(\text{SNR})^{-1}$ , respectively. To calculate the exponent in Eq. (14) with the accuracy  $1/\text{SNR}$  we substitute the expansion (22) into the exponent and arrive at the result:

$$e^{-S[\Psi_{cl}]/Q} = e^{-\frac{(1+4\mu^2/3)x_0^2-2\mu x_0y_0+y_0^2}{QL(1+\mu^2/3)}} \left( 1 - \frac{S_2}{Q} + \left[ \frac{S_2^2}{2Q^2} - \frac{S_3}{Q} \right] + \mathcal{O}(\text{SNR}^{-3/2}) \right). \quad (26)$$

To calculate the normalization factor  $\Lambda$  we also use the

method developed in [18]. First, we change the integration variables in Eq. (15) from  $\tilde{\psi}(z)$  to  $u(z)$  as  $\tilde{\psi}(z) = e^{i\gamma\rho^2z}u(z)$ . Then we expand  $e^{S[\Psi_{cl}(z)+\tilde{\psi}(z)]-S[\Psi_{cl}(z)]}$  in parameter  $Q$ , and find terms of the order of  $Q^0$ ,  $Q^{1/2}$  and  $Q^1$ . After that using the Wick's theorem and correlation function (see Eqs. (98), (103)-(105) in Ref.[18]) we obtain

$$\Lambda = \frac{1}{\pi QL\sqrt{1+\mu^2/3}} \left( 1 + \tilde{\Lambda}_1 + \tilde{\Lambda}_2 + \mathcal{O}\left(\frac{1}{\text{SNR}^{3/2}}\right) \right), \quad (27)$$

where

$$\tilde{\Lambda}_1 = -\frac{3\mu}{5\rho(3+\mu^2)^2} (\mu(15+\mu^2)x_0 - 2(5-\mu^2/3)y_0), \quad (28)$$

$$\tilde{\Lambda}_2 = \frac{\mu^2(11\mu^4+201\mu^2-504)QL}{140(\mu^2+3)^3\rho^2} + \frac{\mu^2}{70(3+\mu^2)^4\rho^2} ((32\mu^6+453\mu^4+8064\mu^2-6237)x_0^2 + 12\mu(4\mu^4+75\mu^2-1323)x_0y_0 - 3(7\mu^6+141\mu^4+1179\mu^2-567)y_0^2). \quad (29)$$

The correction  $\tilde{\Lambda}_1$  was found in Ref. [18], see Eq. (109) therein. This correction contains  $x_0$  and  $y_0$  in the first power, therefore, it is of the order of  $\sqrt{Q}/\rho^2$ . The correction  $\tilde{\Lambda}_2$  contains two different terms. One term is proportional to  $Q/\rho^2$  and another one is the second order homogeneous polynomial in  $x_0$  and  $y_0$ .

Using Eqs. (26) and (27) we obtain the expansion of the conditional PDF:

$$P[Y|X] \approx P_0[Y|X] + \delta P_1[Y|X] + \delta P_2[Y|X], \quad (30)$$

where

$$P_0[Y|X] = \frac{e^{-\frac{(1+4\mu^2/3)x_0^2-2\mu x_0y_0+y_0^2}{QL(1+\mu^2/3)}}}{\pi QL\sqrt{1+\mu^2/3}}, \quad (31)$$

$$\delta P_1[Y|X] = P_0[Y|X] \left( \tilde{\Lambda}_1 - \frac{S_2}{Q} \right), \quad (32)$$

$$\delta P_2[Y|X] = P_0[Y|X] \left( \frac{S_2^2}{2Q^2} - \frac{S_3+S_2\tilde{\Lambda}_1}{Q} + \tilde{\Lambda}_2 \right). \quad (33)$$

One can check that the conditional probability (30) obeys the following important properties:

$$\lim_{Q \rightarrow 0} P[Y|X] = \delta(Y - \Psi_0(L)), \quad (34)$$

$$\lim_{\gamma \rightarrow 0} P[Y|X] = \frac{e^{|Y-X|^2/(QL)}}{\pi QL}, \quad (35)$$

$$\int DY P[Y|X] = 1. \quad (36)$$

The condition (34) is the deterministic limit of  $P[Y|X]$  in the absence of noise. The condition (35) means that our conditional probability transforms to the conditional probability of the linear channel. Note that all found corrections are proportional to the parameter  $\mu = \gamma L\rho^2$ ,

therefore, they disappear when the nonlinearity goes to zero. The last (normalization) condition (36) is the check of correctness of our calculations: one can check that

$$\int DY \delta P_{1,2}[Y|X] = 0, \quad (37)$$

since  $\int DY P_0[Y|X] = 1$ .

### B. PDF $P_{\text{out}}[Y]$ of the output signal

Now we proceed to calculation of the distribution  $P_{\text{out}}[Y]$  of the output signal  $Y$ . Let us consider the integral, see Eq. (8),

$$P_{\text{out}}[Y] = \int \mathcal{D}X P[Y|X] P_X[X], \quad (38)$$

where the input signal PDF  $P_X[X]$  is a smooth function. We assume that the function  $P_X[X]$  changes sufficiently when the variation of the variable  $X$  is of the order of  $\sqrt{P}$ . Since  $QL \ll P \ll (QL^3\gamma^2)^{-1}$  we can calculate the integral (38) by the Laplace's method [25] in the same

manner as we performed the leading order calculation of  $P_{\text{out}}[Y]$ , see Appendix C in Ref. [18]. It is convenient to change the integration variables from  $X = x_1 + iy_1$  to  $\tau = \tau_1 + i\tau_2$ . The substitution has the form:

$$X = \frac{(\sqrt{|Y|^2 - \tau_2^2} - \tau_1)(\sqrt{|Y|^2 - \tau_2^2} - i\tau_2)}{|Y|^2} \times Y \exp \left\{ -i\gamma L \left( \sqrt{|Y|^2 - \tau_2^2} - \tau_1 \right)^2 \right\}. \quad (39)$$

The choice of the substitution (39) is motivated by the fact that at  $\tau = 0$  one has  $X = Ye^{-i\gamma L|Y|^2}$ , and the function  $P[Y|X]$  reaches the maximum at the point  $\tau = 0$ . After the change of variables (39) we perform integration using Laplace's method and obtain:

$$P_{\text{out}}[Y] = P_X[\tilde{Y}] + \delta P_{\text{out}}[\tilde{Y}], \quad (40)$$

where  $\tilde{Y} = Ye^{-i\tilde{\mu}} = \tilde{y}_1 + i\tilde{y}_2$ ,  $\tilde{y}_1 = \text{Re}\tilde{Y}$ ,  $\tilde{y}_2 = \text{Im}\tilde{Y}$ ,  $\tilde{\mu} = \gamma L|Y|^2$ . The correction  $\delta P_{\text{out}}[\tilde{Y}]$  can be expressed through the input signal distribution as follows:

$$\begin{aligned} \delta P_{\text{out}}[\tilde{Y}] = & \frac{\gamma QL^2}{3} \left( (3\tilde{y}_2 - \tilde{\mu}\tilde{y}_1) \frac{\partial P_X[\tilde{Y}]}{\partial \tilde{y}_1} - (3\tilde{y}_1 + \tilde{\mu}\tilde{y}_2) \frac{\partial P_X[\tilde{Y}]}{\partial \tilde{y}_2} - \frac{1}{2}(3(\tilde{y}_1^2 - \tilde{y}_2^2) + 4\tilde{\mu}\tilde{y}_1\tilde{y}_2) \frac{\partial^2 P_X[\tilde{Y}]}{\partial \tilde{y}_1 \partial \tilde{y}_2} \right) + \\ & \frac{QL}{12|Y|^2} \left( (3|Y|^2 + 6\tilde{\mu}\tilde{y}_1\tilde{y}_2 + 4\tilde{\mu}^2\tilde{y}_2^2) \frac{\partial^2 P_X[\tilde{Y}]}{\partial \tilde{y}_1^2} + (3|Y|^2 - 6\tilde{\mu}\tilde{y}_1\tilde{y}_2 + 4\tilde{\mu}^2\tilde{y}_1^2) \frac{\partial^2 P_X[\tilde{Y}]}{\partial \tilde{y}_2^2} \right). \end{aligned} \quad (41)$$

In the polar coordinates  $\tilde{Y} = \tilde{\rho} e^{i\tilde{\phi}}$  the correction  $\delta P_{\text{out}}[\tilde{Y}]$  reads:

$$\begin{aligned} \delta P_{\text{out}}[\tilde{Y}] = & -\frac{\gamma QL^2}{2} \frac{\partial}{\partial \tilde{\phi}} \left( 1 + \tilde{\rho} \frac{\partial}{\partial \tilde{\rho}} - \frac{2}{3} \tilde{\mu} \frac{\partial}{\partial \tilde{\phi}} \right) P_X[\tilde{Y}] + \\ & \frac{QL}{4} \Delta_2 P_X[\tilde{Y}], \end{aligned} \quad (42)$$

where  $\Delta_2$  is Laplace operator. One can see that for an axially symmetric distribution, i.e., when  $P_X[X]$  depends only on  $|X| = \rho$ , the correction (42) has the form  $\delta P_{\text{out}}[\tilde{\rho}] = \frac{QL}{4} \Delta_2 P_X[\rho]$ , which is in agreement with the general (nonperturbative) result, obtained in Ref. [18], see Eq. (32) therein. From Eq. (42) one can see that the first nonzero correction  $\delta P_{\text{out}}[\tilde{Y}]$  to  $P_{\text{out}}[Y]$  has the order  $\mathcal{O}(\gamma QL^2) + \mathcal{O}(QL/\rho^2)$ , since  $|Y| \sim |X|$ . Note that the validity of our approximation (41) and the possibility to use Laplace's method are justified by that the power  $P$  is from the intermediate power region  $QL \ll P \ll (\gamma^2 L^3 Q)^{-1}$ : see the detailed explanation in [18], Appendix C.

### III. CALCULATION OF ENTROPIES

To calculate the conditional entropy with the accuracy  $1/\text{SNR}$  we substitute the conditional PDF (30) to Eq. (6) and obtain:

$$\begin{aligned} H[Y|X] \approx & - \int \mathcal{D}X \mathcal{D}Y P_X[X] \left( \log P_0[Y|X] \times \right. \\ & (P_0[Y|X] + \delta P_1[Y|X] + \delta P_2[Y|X]) + \\ & \left. \frac{\delta P_1^2[Y|X]}{2P_0[Y|X]} \right). \end{aligned} \quad (43)$$

To obtain Eq. (43) we used the consequence (37) of the normalization condition for the function  $P[Y|X]$ . The direct integration over  $Y$  in Eq. (43) gives

$$H[Y|X] \approx H_0[Y|X] + \delta H[Y|X], \quad (44)$$



where

$$\begin{aligned} H_0[Y|X] &= 1 + \log(\pi QL) + \\ &\quad \frac{1}{2} \int \mathcal{D}X P_X[X] \log \left( 1 + \frac{\mu^2}{3} \right), \quad (45) \\ \delta H[Y|X] &= QL \int \mathcal{D}X P_X[X] \times \\ &\quad \frac{\mu^2 (-13\mu^4 + 255\mu^2 + 450)}{150 (3 + \mu^2)^3 |X|^2}. \quad (46) \end{aligned}$$

The leading in  $1/\text{SNR}$  term (45) for the conditional entropy was obtained in Ref. [18]. Here we obtain the correction (46). One can see that the correction (46) is proportional to  $Q$  and  $\gamma^2$ . Therefore, it vanishes for the linear case  $\gamma = 0$ .

To calculate the output signal entropy (7) we substitute  $P_{\text{out}}[Y]$ , Eq. (40), to Eq. (7) and obtain

$$\begin{aligned} H[Y] &= - \int \mathcal{D}Y \left( P_X[\tilde{Y}] \log P_X[\tilde{Y}] + \right. \\ &\quad \left. \delta P_{\text{out}}[\tilde{Y}] \left\{ 1 + \log P_X[\tilde{Y}] \right\} \right). \quad (47) \end{aligned}$$

Let us note that  $\mathcal{D}Y = dy_1 dy_2 = \mathcal{D}\tilde{Y} = d\tilde{y}_1 d\tilde{y}_2$ . The first term in the right-hand side of Eq. (47) coincides with the leading order contribution obtained in Ref. [18], see Eq. (39) therein. That is nothing else but the input signal entropy  $H[X]$ . The second term in the right-hand side of Eq. (47) is proportional to parameter  $QL$ . We can omit the unity in the curly brackets in Eq. (47) owing to the normalization condition for  $P_{\text{out}}[Y] = P_X[\tilde{Y}] + \delta P_{\text{out}}[\tilde{Y}]$ :  $\int \mathcal{D}Y P_{\text{out}}[Y] = 1$ , and therefore,  $\int \mathcal{D}\tilde{Y} \delta P_{\text{out}}[\tilde{Y}] = 0$ .

#### IV. OPTIMAL INPUT SIGNAL DISTRIBUTION

To calculate the channel capacity (2) we should find the optimal input signal distribution  $P_{\text{opt}}[X]$  which is defined as

$$C = \max_{P_X[X]} I_{P_X[X]} = I_{P_{\text{opt}}[X]}. \quad (48)$$

To find the optimal input signal distribution  $P_{\text{opt}}[X]$  normalized to unity and with the fixed average power  $P$  we solve the variational problem, see Section III in Ref. [18]:

$$\delta J[P_X, \lambda_1, \lambda_2] = 0 \quad (49)$$

with the functional  $J[P_X, \lambda_1, \lambda_2]$  that reads

$$\begin{aligned} J[P_X, \lambda_1, \lambda_2] &= H[Y] - H[Y|X] - \\ &\quad \lambda_1 \left( \int \mathcal{D}X P_X[X] - 1 \right) - \\ &\quad \lambda_2 \left( \int \mathcal{D}X P_X[X] |X|^2 - P \right), \quad (50) \end{aligned}$$

where  $\lambda_{1,2}$  are the Lagrangian coefficients, at that  $H[Y|X]$  and  $H[Y]$  are given by Eqs. (44) and (47), respectively. The solution of the equation (49) in the leading order in the parameter  $Q$  was found in Ref. [18]:

$$P_{\text{opt}}^{(0)}[X] = N_0 \frac{\exp \{ -\lambda_0 |X|^2 \}}{\sqrt{1 + \mu^2/3}}, \quad (51)$$

where  $\mu = \gamma L |X|^2$ . The functions  $N_0 = N_0(P)$  and  $\lambda_0 = \lambda_0(P)$  are determined from the conditions:

$$\int \mathcal{D}X P_{\text{opt}}^{(0)}[X] = 2\pi N_0 \int_0^\infty \frac{d\rho \rho e^{-\lambda_0 \rho^2}}{\sqrt{1 + \gamma^2 L^2 \rho^4/3}} = 1, \quad (52)$$

$$\int \mathcal{D}X P_{\text{opt}}^{(0)}[X] |X|^2 = 2\pi N_0 \int_0^\infty \frac{d\rho \rho^3 e^{-\lambda_0 \rho^2}}{\sqrt{1 + \gamma^2 L^2 \rho^4/3}} = P. \quad (53)$$

The solutions  $\lambda_0$  and  $N_0$  can be found numerically for any arbitrary case. Note that the products  $\lambda_0 P$  and  $N_0 P$  are the functions of dimensionless nonlinearity parameter  $\tilde{\gamma} = \gamma PL/\sqrt{3}$  only. For the case of small nonlinearity parameter  $\tilde{\gamma}$  the solutions have the form:

$$\lambda_0(P) = \frac{1}{P} (1 - 2\tilde{\gamma}^2), \quad N_0(P) = \frac{1}{\pi P} (1 - \tilde{\gamma}^2). \quad (54)$$

In the case of sufficiently large parameter  $\tilde{\gamma}$  such as  $\log \tilde{\gamma} \gg 1$  using the results of Ref. [18] one can obtain the following asymptotics:

$$\lambda_0 \approx \frac{1 - \log \log(c\tilde{\gamma}) / \log(c\tilde{\gamma})}{P \log(c\tilde{\gamma})}, \quad (55)$$

$$N_0 \approx \frac{\tilde{\gamma}}{\pi P} \log^{-1} [c\tilde{\gamma} / (\lambda_0 P)], \quad (56)$$

where  $c = 2e^{-\gamma E}$  and the accuracy of asymptotic estimates (55) and (56) is  $\mathcal{O}(\log^{-2}(\tilde{\gamma}))$ .

To calculate the corrections of the order of  $Q$  to the solution (51) we substitute the optimal input PDF in the following form

$$P_{\text{opt}}[X] \approx P_{\text{opt}}^{(0)}[X] + P_{\text{opt}}^{(1)}[X] \quad (57)$$

to Eq. (50), where  $P_{\text{opt}}^{(0)}[X]$  is defined in Eq. (51) and  $P_{\text{opt}}^{(1)}[X]$  is the first correction proportional to  $Q$ . Then we keep terms which are proportional to  $Q$  and obtain:

$$P_{\text{opt}}^{(1)}[X] = QL \left( -\lambda_0^2 |X|^2 + \frac{2\lambda_0}{1 + \mu^2/3} + \mu^2 \frac{(-137\mu^4 + 1095\mu^2 + 4950)}{4050|X|^2(1 + \mu^2/3)^3} \right) P_{\text{opt}}^{(0)}[X] - (\delta\lambda_1 + \delta\lambda_2 |X|^2) P_{\text{opt}}^{(0)}[X], \quad (58)$$

$$\delta\lambda_1 = \frac{QL/P}{750\tilde{\gamma}^2(\tilde{\gamma}^2(P\lambda_0 - 1) + P\lambda_0 - \pi PN_0)} \left\{ 16P^4\lambda_0^2(\lambda_0 - \pi N_0)^2 + \tilde{\gamma}^4 \left( P\lambda_0(-1370 + 1379P\lambda_0) - 428\pi PN_0 \right) + \right. \\ \left. \tilde{\gamma}^2 \left( P^2\lambda_0^2(685 + 16P\lambda_0[P\lambda_0 - 4]) + \pi P^2 N_0\lambda_0(48P\lambda_0 - 257) - 428\pi^2 P^2 N_0^2 \right) \right\}, \quad (59)$$

$$\delta\lambda_2 = \frac{\lambda_0 QL/P}{750(\tilde{\gamma}^2(P\lambda_0 - 1) + P\lambda_0 - \pi PN_0)} \left\{ \tilde{\gamma}^2 \left( 685 - 347P\lambda_0(1 + P\lambda_0) + 428\pi PN_0 \right) + \right. \\ \left. P\lambda_0 \left( 315\pi PN_0 - P\lambda_0(299 + 16\pi PN_0) \right) \right\}. \quad (60)$$

Since  $P_{\text{opt}}^{(0)}[X]$  obeys the normalization conditions (52) and (53), therefore, the correction (58) must obey the following two conditions:

$$\int \mathcal{D}X P_{\text{opt}}^{(1)}[X] = 0, \quad (61)$$

$$\int \mathcal{D}X |X|^2 P_{\text{opt}}^{(1)}[X] = 0. \quad (62)$$

One can check that for  $\delta\lambda_{1,2}$  from Eqs. (59), (60) these conditions are fulfilled.

## V. CAPACITY IN THE NEXT-TO-LEADING ORDER

To calculate the channel capacity up to the terms proportional to  $Q$  we substitute the optimal input signal distribution in the form (57) to the mutual information (5) and obtain

$$C = C_0 + \Delta C, \quad (63)$$

where the leading order contribution  $C_0$  reads, see Eq. (51) in Ref. [18]:

$$C_0 = \log(\text{SNR}) + \lambda_0 P - \log(\pi N_0 P) - 1, \quad (64)$$

and the required next-to-leading correction has the form

$$\Delta C = \frac{1}{\text{SNR}} \left\{ \pi N_0 P \left[ \frac{214}{375} - \frac{8}{375} \left( \frac{\lambda_0 P}{\tilde{\gamma}} \right)^2 \right] + \right. \\ \left. + \lambda_0 P \left[ \frac{137}{150} + \frac{8}{375} \left( \frac{\lambda_0 P}{\tilde{\gamma}} \right)^2 \right] - \frac{347}{750} (\lambda_0 P)^2 \right\}. \quad (65)$$

The term  $\Delta C$  is the first nonvanishing correction to the capacity. One can check that for small parameter  $\gamma L^2 Q \ll 1$  the correction (65) is always small. Indeed, the expression in the curly bracket in Eq. (65) divided by  $\tilde{\gamma}$  is limited for all  $\tilde{\gamma}$ . This correction can be calculated

numerically for arbitrary parameter  $\tilde{\gamma}$ , and analytically for small and large  $\tilde{\gamma}$ .

First, let us consider the correction at small nonlinearity. We substitute the parameters  $\lambda_0$  and  $N_0$  in the form (54) and obtain:

$$\Delta C \approx \frac{1}{\text{SNR}} - \frac{1}{\text{SNR}} \frac{\tilde{\gamma}^2}{3}. \quad (66)$$

Using this result and expansion of the  $C_0$  at small nonlinearity, see Eq. (53) in Ref. [18], we can write the capacity within our accuracy in the form:

$$C \approx \log(1 + \text{SNR}) - \tilde{\gamma}^2 - \frac{1}{\text{SNR}} \frac{\tilde{\gamma}^2}{3}. \quad (67)$$

One can see that the nonlinear correction is negative for small  $\tilde{\gamma}$  and it reduces the result for the linear channel.

More interesting is to consider the correction to the capacity at large power  $P$ . For the case  $\log(\gamma LP) \gg 1$  and  $P \ll (\gamma^2 QL^3)^{-1}$  we have the simple representation:

$$\Delta C \approx \frac{1}{\text{SNR}} \frac{214}{375} \pi N_0 P. \quad (68)$$

Using the asymptotic formulae (56), (55) for quantity  $N_0$  we arrive at the expression

$$\Delta C \approx \frac{\gamma L^2 Q}{\sqrt{3}} \times \frac{214}{375} \left( \log(c\tilde{\gamma} \log(c\tilde{\gamma})) + \frac{\log \log(c\tilde{\gamma})}{\log(c\tilde{\gamma})} \right)^{-1}. \quad (69)$$

We take notice that this correction is suppressed as  $\gamma L^2 Q$  instead of  $1/\text{SNR} = QL/P$  and it decreases as  $1/\log \tilde{\gamma}$  at large  $\tilde{\gamma}$ . For large  $\tilde{\gamma}$  the correction (69) is positive, therefore, it enhances the capacity.

For the further consideration of the correction it is convenient to subtract the term  $\frac{1}{\text{SNR}}$ , which corresponds to the expansion of the Shannon's logarithm (1) at large SNR, from the correction (65):

$$\Delta C' = \Delta C - \frac{1}{\text{SNR}}. \quad (70)$$

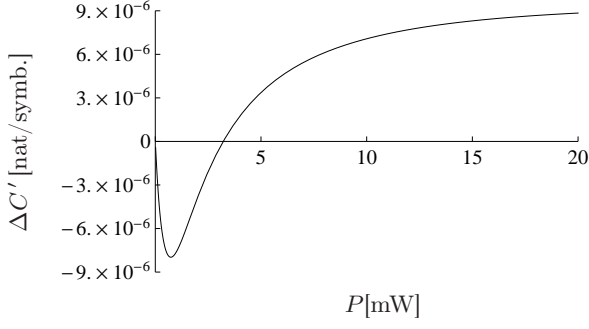


Figure 1: The correction  $\Delta C'$ , see Eq. (70), as a function of power  $P$  for the parameters  $Q = 1.5 \times 10^{-7} \text{ mW km}^{-1}$ ,  $\gamma = 1.3 \times 10^{-3} \text{ mW}^{-1} \text{ km}^{-1}$ ,  $L = 1000 \text{ km}$ .

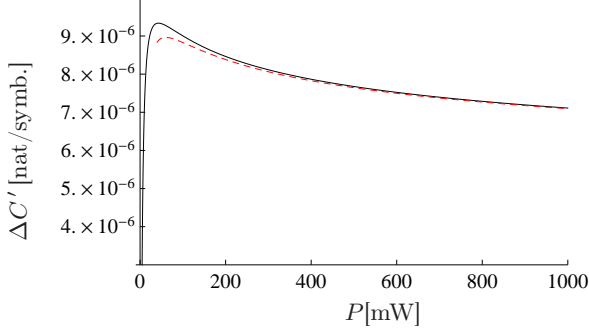


Figure 2: The correction  $\Delta C'$ , see Eq. (70), as a function of power  $P$  for the parameters  $Q = 1.5 \times 10^{-7} \text{ mW km}^{-1}$ ,  $\gamma = 1.3 \times 10^{-3} \text{ mW}^{-1} \text{ km}^{-1}$ ,  $L = 1000 \text{ km}$ . The solid black line corresponds to the exact expression obtained using Eq. (65). The red dashed line corresponds to the asymptotics obtained using Eq. (69).

The correction  $\Delta C'$  is convenient for analysis since it is regular function for all range of signal power  $P$ . Let us consider the correction  $\Delta C'$  for the parameters  $Q = 1.5 \times 10^{-7} \text{ mW km}^{-1}$ ,  $\gamma = 1.3 \times 10^{-3} \text{ mW}^{-1} \text{ km}^{-1}$ ,  $L = 1000 \text{ km}$  which can be realized in experiment, see [17]. Note that for chosen parameters the intermediate power region  $QL \ll P \ll (\gamma^2 L^3 Q)^{-1}$  is extremely broad:

$$1.5 \times 10^{-4} \text{ mW} \ll P \ll 3.89 \times 10^3 \text{ mW}. \quad (71)$$

For these parameters the correction  $\Delta C'$  and its asymptotics are plotted in Fig. 1 and Fig. 2 in the case of moderate and large power  $P$ , respectively. One can see that the correction  $\Delta C'$  reaches the minimum  $-7.97 \times 10^{-6} \text{ nat/symb}$  at  $P \approx 0.73 \text{ mW}$  (it corresponds to  $\tilde{\gamma}_{\min} \approx 0.55$ ), see Fig. 1, and the maximum  $9.35 \times 10^{-6} \text{ nat/symb}$  at  $P \approx 43.4 \text{ mW}$  (it corresponds to  $\tilde{\gamma}_{\max} = 32.82$ ), see Fig. 2. In the wide power region  $P \gg (\gamma L)^{-1} \approx 0.76 \text{ mW}$  and  $P \ll (\gamma^2 L^3 Q)^{-1} \approx 4 \times 10^3 \text{ mW}$ , see Fig. 2, the correction  $\Delta C'$  is almost constant  $\Delta C' \approx 4 \times 10^{-2} \times \gamma Q L^2 \approx 8 \times 10^{-6} \text{ nat/symb}$ .

## VI. CONCLUSION

We calculated the first nonzero corrections to the optimal input signal distribution  $P_{\text{opt}}$ , the output signal distribution  $P_{\text{out}}$ , and channel capacity  $C$  for the nondispersive nonlinear channel in the case when the noise power  $QL$  is much less than the signal power  $P$ . These corrections are proportional to the noise power  $QL$ . We demonstrated that the correction  $\Delta C$  to the channel capacity is small in the intermediate power region  $QL \ll P \ll (\gamma^2 L^3 Q)^{-1}$ . At large signal power  $P$ ,  $(\gamma L)^{-1} \ll P \ll (\gamma^2 L^3 Q)^{-1}$ , the correction  $\Delta C$  is the positive decreasing function. We stress that  $\Delta C$  is suppressed as  $1/\text{SNR} = QL/P$  for small parameter  $\tilde{\gamma} = \gamma LP/\sqrt{3}$  in comparison with the leading order contribution, and it is suppressed as  $\gamma L^2 Q$  decreasing as  $1/\log \tilde{\gamma}$  at large  $\tilde{\gamma}$ . The calculation of the channel capacity  $C_0$  was carried out in assumption that the parameter  $\gamma^2 L^3 Q P \ll 1$ , or  $P \ll (\gamma^2 L^3 Q)^{-1}$ . Since among the corrections proportional to  $QL$  there are no corrections of the order of  $\gamma^2 L^3 Q P$  at large  $P$ , we can expect that the next correction which contains power  $P$  should be of the order of  $(\gamma^2 L^3 Q P)^2$ , see Ref. [18]. Therefore, the applicability region at large  $P$  for the channel capacity  $C_0$  is determined by the condition  $(\gamma^2 L^3 Q P)^2 \ll 1$ . For the given small parameter  $\gamma^2 L^3 Q P$  this condition extends the applicability region for the channel capacity  $C_0$ .

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- [1] C. Shannon, *A mathematical theory of communication*, Bell Syst. Tech. J. **27**, 379 (1948); **27**, 623 (1948).
- [2] P. P. Mitra and J. B. Stark, *Nature* **411**, 1027 (2001).
- [3] E. E. Narimanov and P. Mitra, *J. Lightwave Technol.* **20**, 530 (2002).
- [4] J. M. Kahn and K.-P. Ho, *IEEE J. Sel. Topics Quant. Electron.* **10**, 259 (2004).
- [5] R.-J. Essiambre, G. J. Foschini, G. Kramer, and P. J. Winzer, *Phys. Rev. Lett.* **101**, 163901 (2008).
- [6] R.-J. Essiambre, G. Kramer, P. J. Winzer, G. J. Foschini,

- and B. Goebel, *J. of Lightwave Technol.* **28**, 662 (2010).
- [7] R. Killey and C. Behrens, *J. Mod. Opt.* **58**, 1 (2011).
- [8] E. Agrell, A. Alvarado, G. Durisi, and M. Karlsson, *IEEE/OSA J. Lightwave Technol.* **32**, 2862 (2014).
- [9] M. A. Sorokina and S. K. Turitsyn, *Nat. Comm.* **5**, 3861 (2014).
- [10] I. S. Terekhov, A. V. Reznichenko, and S. K. Turitsyn, arXiv:1607.01184.
- [11] I. S. Terekhov, A. V. Reznichenko, and S. K. Turitsyn, arXiv:1411.7477.



- [12] K. S. Turitsyn, S. K. Turitsyn, Opt. Lett. **37**, 3600 (2012).
- [13] A. Mecozzi, J. Lightwave Technol. **12**, 1993 (1994).
- [14] A. Mecozzi and M. Shtaif, IEEE Photonics Technol. Lett. **13**, 1029 (2001).
- [15] J. Tang, J. Lightwave Technol. **19**, 1104 (2001).
- [16] K.S. Turitsyn, S.A. Derevyanko, I.V. Yurkevich, and S.K. Turitsyn, Phys. Rev. Lett. **91**, 203901 (2003).
- [17] M. I. Yousefi and F. R. Kschischang, IEEE Trans. on Information Theory **57**, 7522 (2011).
- [18] I. S. Terekhov, A. V. Reznichenko, Ya. A. Kharkov, and S. K. Turitsyn, arXiv:1508.05774.
- [19] E. Agrell, IEEE Trans. Commun. **63**, 738 (2015).
- [20] E. Agrell, *Nonlinear Fiber Capacity*, presented at Eur. Conf. Opt. Commun. London U.K., paper We.4.D.3 (2013).
- [21] I. S. Terekhov, S. S. Vergeles, and S. K. Turitsyn, Phys. Rev. Lett. **113**, 230602 (2014).
- [22] P. C. Martin, E. D. Siggia, and H. A. Rose, Phys. Rev. A **8**, 423 (1973).
- [23] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena* (Oxford University Press, Oxford, 2002).
- [24] R. P. Feynman and A. R. Hibbs, *Quantum mechanics and path integrals* (McGraw-Hill Book Company, New York, 1965).
- [25] M. A. Lavrentiev and B. V. Shabat, *Method of Complex Function Theory*. (Nauka, Moscow, 1987, in Russian); M. Lavrentiev and B. Chabot, *Methodes de la Theorie des fonctions d'une variable complexe* (Mir, Moscou, 1977, in French).